# Rank 3 representations of Painlevé systems

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# Basic notations

Fix the following data:

- $X$ a smooth projective curve (later $X = \mathbb{C}P^1$)
- $\mathbb{C}P^1 = \operatorname{Spec}\mathbb{C}[z] \cup \operatorname{Spec}\mathbb{C}[w]$ with $w = z^{-1}$
- $r \geq 2$ (rank of the vector bundle over $X$), $G = \mathrm{GL}(r, \mathbb{C})$
- $D = \sum m_i p_i$ effective divisor on $X$ (later with $|D| = 3$)
- $K_X(D)$ twisted canonical bundle

# Moduli space of meromorphic Higgs bundles

A rank $r$ meromorphic Higgs bundle on $X$ consists of the data

- a holomorphic rank $r$ vector bundle $\mathcal{E}$ over $X$
- a meromorphic Higgs field $\theta \in H^0(X, End(\mathcal{E}) \otimes K_X(D))$

$\mathcal{M}_{r,d}(X, D)$: moduli space of isomorphism classes of stable meromorphic Higgs bundles of given rank $r$ and degree $d$ over $X$

Proposition (E. Markman '94)

- $\mathcal{M}_{r,d}(X, D)$ *is a holomorphic Poisson manifold*
- *symplectic leaves of maximal rank of* $\mathcal{M}$ *are given by fixing the gauge equivalence class of its irregular part at* $D$ *and the corresponding eigenvalues of the residue of* $\theta$ *(c.f. next page).*

The cases when the maximal symplectic leaves are of complex dimension 2, have been listed by P. Boalch.

# Holomorphic symplectic leaves

In the coordinate chart $z - p_i$ and in a local trivialization of $\mathcal{E}$ at $p_i$:

$$\theta = \sum_{k=-m_i}^{\infty} A_k (z - p_i)^k \mathrm{d}z,$$

- The irregular part of $\theta$ is $\int \sum_{k=-m_i}^{-2} A_k (z - p_i)^k \mathrm{d}z$
- the residue of $\theta$ at $p_i \in D$ is $A_{-1}(z - p_i)^{-1}\mathrm{d}z$.

To get symplectic leaves, we need to fix the eigenvalues of the irregular part and the adjoint orbit of the residue in each eigenspace. Moreover, we may choose Dolbeault parabolic weights $\{\alpha^j_{p_i}\}$ at all $p_i \in D$.

## Definition

$\mathcal{M}_{Dol}$: Dolbeault moduli space of rank $r$ meromorphic, parabolic, $\alpha$-stable Higgs bundles over $\mathbb{C}P^1$ with the above fixed data.

# Hitchin–Kobayashi correspondence

$h$ is a Hermitian-Einstein metric in $\mathcal{E}$ if the connection associated to $(\mathcal{E}, \theta, h)$:

$$\nabla = D_h^+ + \theta + \theta^{\dagger_h}$$

is integrable on $X \setminus D$. Here, $D_h^+ = \bar{\partial}_{\mathcal{E}} + \partial_h$ stands for the Chern connection associated to $\bar{\partial}_{\mathcal{E}}, h$.
At each $p_i$, with respect to a frame compatible with the parabolic filtration, we require the asymptotic behavior

$$h \approx \mathrm{diag}(|z|^{2\alpha_{p_i}^j} |\ln |z|^2|^k),$$

$k =$ the weight induced by the nilpotent part of the residue.

Theorem (Biquard–Boalch'04, Mochizuki'11)

*A Hermitian-Einstein metric for $(\mathcal{E}, \theta)$ with the given behavior exists if and only if $(\mathcal{E}, \theta)$ is $\alpha$-polystable.*

# Meromorphic connections

## Definition

A meromorphic connection on a holomorphic vector bundle $E$ over $X$ is a $\mathbb{C}$-linear map

$$\nabla^{1,0}\colon E \to E \otimes K_X(D)$$

satisfying Leibniz' rule.

Let $\{\beta^j_{p_i}\}$ denote the de Rham parabolic weights at $p_i$, and $\mu_i$ the eigenvalues of the residues of $\nabla^{1,0}$.

Let $\mathcal{D}_X$ be the sheaf of analytic differential operators on $X$. A left module $\mathbb{M}$ over $\mathcal{D}_X$ is holonomic if it is finitely generated and torsion. Then a meromorphic connection induces a holonomic $\mathcal{D}_X$-module structure on the sheaf $E$:

$$\partial_t(e) = \nabla^{1,0}_{\partial_t} e.$$

# Formal decomposition of holonomic $\mathcal{D}$-modules

For $c \in C = \mathrm{Sing}(\mathbb{M})$, we set $\mathbb{M}_c = \mathbb{C}[\![t]\!]\langle\partial_t\rangle \otimes_{\mathbb{C}[t]\langle\partial_t\rangle} \mathbb{M}$, with $t$ a local chart centered at $c$.

## Theorem (Hukuhara–Levelt–Turrittin)

*Let $\mathbb{M}$ be a holonomic $\mathcal{D}_X$-module induced by a meromorphic connection with pole at $\mathrm{Spec}\,\mathbb{C}[t]/(t)$. Then $\exists\ d \in \mathbb{N}$ and a finite Galois extension $K = \mathbb{C}(\!(u,t)\!)/(u^d - t)$ of $\mathbb{C}(\!(t)\!)$ s.t.*

$$K \otimes_{\mathbb{C}[t]} \mathbb{M} \cong \bigoplus_{q \in K/\mathbb{C}[\![u]\!]} (\mathbb{C}[\![u]\!], \mathrm{d} + \mathrm{d}q) \otimes_{\mathbb{C}[\![u]\!]} \mathcal{F}_q$$

*for some locally free $\mathbb{C}[\![u]\!]$-modules $\mathcal{F}_q$ endowed with a regular singular connection.*

# Irregular type, slope, etc.

## Definition

- The irregular type of $\mathbb{M}_c$ is the formal sum $\sum_q \mathrm{rk}(\mathcal{F}_q)q$. It can be thought of as a marked finite cover of $S^1$.
- The slope of $q$ is $\lambda(q) = \frac{\deg_u q}{d}$.
- The Swan conductor (or irregularity) of $\mathbb{M}_c$ is $\mathrm{Sw}(\mathbb{M}_c) = \sum_q \mathrm{rk}(\mathcal{F}_q)\lambda(q)$.
- The exponential torus of an irregular type $\sum m_q q$ is $\left(\prod_q T^{m_q}\right)/\mathbb{C}^\times$, where $T^{m_q} \subset \mathrm{GL}(m_q, \mathbb{C})$ is the standard maximal torus.
- The formal monodromy of $\mathbb{M}_c$ is the product $H \in \prod_q \mathrm{GL}(m_q, \mathbb{C})$ of the monodromies of $\mathcal{F}_q$.

# Irregular de Rham moduli space

**Definition**

$\mathcal{M}_{dR}$: de Rham moduli space of $\beta$-stable integrable connections with irregular singularities of fixed irregular types and residues at $D$.

The transformation of the irregular types from Dolbeault to de Rham local forms is $\frac{1}{2}q_i \mapsto q_i$.
The transformation of the $\nu_i$ eigenvalues of the residues and the $\alpha_i$ parabolic weights from the Dolbeault side, obeys Simpson's formulas:

$$\alpha_i = \operatorname{Re}\mu_i, \quad \nu_i = \frac{\mu_i - \beta_i}{2}.$$

# Non-abelian Hodge moduli space

## Theorem (Biquard–Boalch'04)

*For fixed irregular types $\sum m_q q$ and adjoint orbits $\mathcal{O}_i$, there exist moduli spaces $\mathcal{M}_{Dol}$ of gauge equivalence classes of $\alpha$-stable meromorphic Higgs bundles having given irregular type, and with residue at $p_i$ belonging to $\mathcal{O}_i$. The spaces $\mathcal{M}_{Dol}$ are complete hyperKähler manifolds. Moreover, endowed with another complex structure $J$ of the hyperKähler family, the spaces $\mathcal{M}_{Dol}$ are $\mathbb{R}$-analytically isomorphic to moduli spaces $\mathcal{M}_{dR}$ of $\beta$-stable integrable connections with irregular singularities of fixed irregular types and residues in given adjoint orbits. In particular, there exist diffeomorphisms*

$$NAH\colon \mathcal{M}_{Dol} \longrightarrow \mathcal{M}_{dR}.$$

# Cases related to root system $\widetilde{E}_6$

The case of interest to us is the one related to the root system $\widetilde{E}_6$, that has numerical data $r = 3$ and $\text{length}(D) = 3$.
Three different partitions realize the possible divisors $D$:
$(1, 1, 1), (2, 1)$ and $(3)$.
The first one will be referred to as the logarithmic case, and the other two as the irregular ones.
In both irregular cases, the Jordan type of the leading-order coefficient of $\theta$ may belong to three combinatorially different adjoint orbits

- semisimple orbits (untwisted),
- orbits with a Jordan block of size 2 and a Jordan block of size 1 (minimally twisted),
- and finally orbits with a Jordan block of size 3 (maximally twisted).

All together 7 combinatorical cases.

# Joshi–Kitaev–Treharne systems

The 6 irregular cases (related to the root system $\widetilde{E}_6$) of rank 3 connections, were studied by Joshi, Kitaev, and Treharne'07,'09, in order to give the $3 \times 3$ Lax representations of the Painlevé equations.

We will refer to these as $JKT*$-cases, where

$$* \in \{VI, V, IVa, IVb, II, I\}.$$

- Cases $JKTVI, JKTV, JKTIVa$ correspond to $D = \{0\} + 2 \cdot \{\infty\}$.
- Cases $JKTIVb, JKTII, JKTI$ correspond to $D = 3 \cdot \{\infty\}$.

The corresponding de Rham and Dolbeault moduli spaces are denoted by $\mathcal{M}_{dR}^{JKT*}$ and $\mathcal{M}_{Dol}^{JKT*}$

# Joshi–Kitaev–Treharne systems

| Summary of the investigated cases | | |
|---|---|---|
| *JKT* system | polar divisor of $\nabla$ | type of irregular singularity at $\infty$ |
| *JKTVI* | $D = \{0\} + 2\{\infty\}$ | untwisted |
| *JKTV* | $D = \{0\} + 2\{\infty\}$ | minimally twisted |
| *JKTIVa* | $D = \{0\} + 2\{\infty\}$ | maximally twisted |
| *JKTIVb* | $D = 3\{\infty\}$ | untwisted |
| *JKTII* | $D = 3\{\infty\}$ | minimally twisted |
| *JKTI* | $D = 3\{\infty\}$ | maximally twisted |

# Example - *JKTII* case

$D = 3 \cdot \{\infty\}$, such that the irregular singularity $p_0 = \infty$ is minimally twisted. Local normal form of the meromorphic Higgs field at $\infty$:

$$\theta = \left[ \begin{pmatrix} a_0 & 1 & 0 \\ 0 & a_0 & 0 \\ 0 & 0 & a_1 \end{pmatrix} w^{-3} + \begin{pmatrix} 0 & 0 & 0 \\ b_0 & b_1 & 0 \\ 0 & 0 & b_2 \end{pmatrix} w^{-2} + \begin{pmatrix} 0 & 0 & 0 \\ c_0 & c_1 & 0 \\ 0 & 0 & c_2 \end{pmatrix} w^{-1} + O(1) \right] \otimes \mathrm{d}w,$$

where $a_0, a_1, b_i, c_i \in \mathbb{C}$ $(i = 0, 1, 2)$ are fixed, and $a_0 \neq a_1$.
The irregular type of $\theta$ is given by the Laurent polynomials:

$$\frac{1}{2} q_{0,1} = q_{\pm} = \lambda_1 w^{-1/2} + \lambda_2 w^{-1} + \lambda_3 w^{-3/2} - \frac{1}{2} a_0 w^{-2}$$

$$\frac{1}{2} q_2 = -b_2 w^{-1} - \frac{1}{2} a_1 w^{-2}.$$

with corresponding residue eigenvalues $0, c_1, c_2$. The coefficients $\lambda_i$ $(1 \leq i \leq 3)$ depend algebraically on $b_i, c_i$.

# Example - *JKTII* (cont'd)

The de Rham local form of $\nabla$ at $\infty$ is defined as

$$\nabla = \mathrm{d} + \mathrm{d}Q_i + \Lambda_i \frac{\mathrm{d}z}{z} + O(1)\,\mathrm{d}z,$$

The de Rham irregular type:

$$q_{0,1} = 2(\lambda_1 w^{-1/2} + \lambda_2 w^{-1} + \lambda_3 w^{-3/2} - \frac{1}{2} a_0 w^{-2})$$

$$q_2 = 2(-b_2 w^{-1} - \frac{1}{2} a_1 w^{-2}).$$

The corresponding de Rham local form:

$$\nabla = \mathrm{d} + \left[ \begin{pmatrix} 2a_0 & 1 & 0 \\ 0 & 2a_0 & 0 \\ 0 & 0 & 2a_1 \end{pmatrix} w^{-3} + \begin{pmatrix} 0 & 0 & 0 \\ b_0' & b_1' & 0 \\ 0 & 0 & b_2' \end{pmatrix} w^{-2} + \begin{pmatrix} 0 & 0 & 0 \\ c_0' & c_1' & 0 \\ 0 & 0 & c_2' \end{pmatrix} w^{-1} + O(1) \right] \otimes \mathrm{d}w$$

where $b_i', c_i'$ correspond to $b_i, c_i$ via Simpson's transformation.

# Stokes local systems – irregular types

$G = \mathrm{GL}(3, \mathbb{C})$
$T \subset G$ maximal torus with Lie algebra $\mathfrak{t}$

$$Q_i = \sum_{k=-m_i}^{-1} A'_{k-1}(z - p_i)^k,$$

$Q_i$ irregular type singles out a subgroup

$$H_i = \{g \in G | \ \mathrm{Ad}_g(A'_j) = A'_j \text{ for all } j \geq 1\} \subset G$$

called its centralizer.
$\mathcal{R} \subset \mathfrak{t}^\vee$: set of roots of the Lie algebra $\mathfrak{g}$ of $G$ w.r.t. $\mathfrak{t}$
$\widetilde{X}$: real oriented blow-up of $X$ at $p_i$
$\partial_i$: fiber of $\widetilde{X}$ over $p_i$ (boundary circle)
Elements of $\partial_i$ are called directions at $p_i$, and are denoted by $d$.

# Stokes local systems – Stokes directions and subgroups

## Definition

A direction $d$ is called a singular direction supported by $\alpha \in \mathcal{R}$ if it is the tangent direction to one of the components of the curve

$$\mathrm{Im}(\alpha \circ Q(z_i)) = 0, \quad \mathrm{Re}(\alpha \circ Q(z_i)) < 0.$$

$\mathcal{R}(d) = \{\alpha | \alpha \text{ supports singular direction } d\}$
$\mathbb{S}\mathrm{to}_d(Q_i) \subset G$: closed subgroup corresponding to the Lie subalgebra spanned by $\mathfrak{g}_\alpha$ for all $\alpha \in \mathcal{R}(d)$.

$$\mathbb{S}\mathrm{to}(Q_i) = \prod_{d \in \partial_i} \mathbb{S}\mathrm{to}_d(Q_i),$$

# Stokes representations

For all $i$, fix a base point $b_i$ near $p_i \rightsquigarrow$ fundamental groupoid $\Pi$ of the irregular curve. For all singular directions $d$ at $p_i$, make a puncture on $\widetilde{X}$ "close to" $d \in \partial_i$.

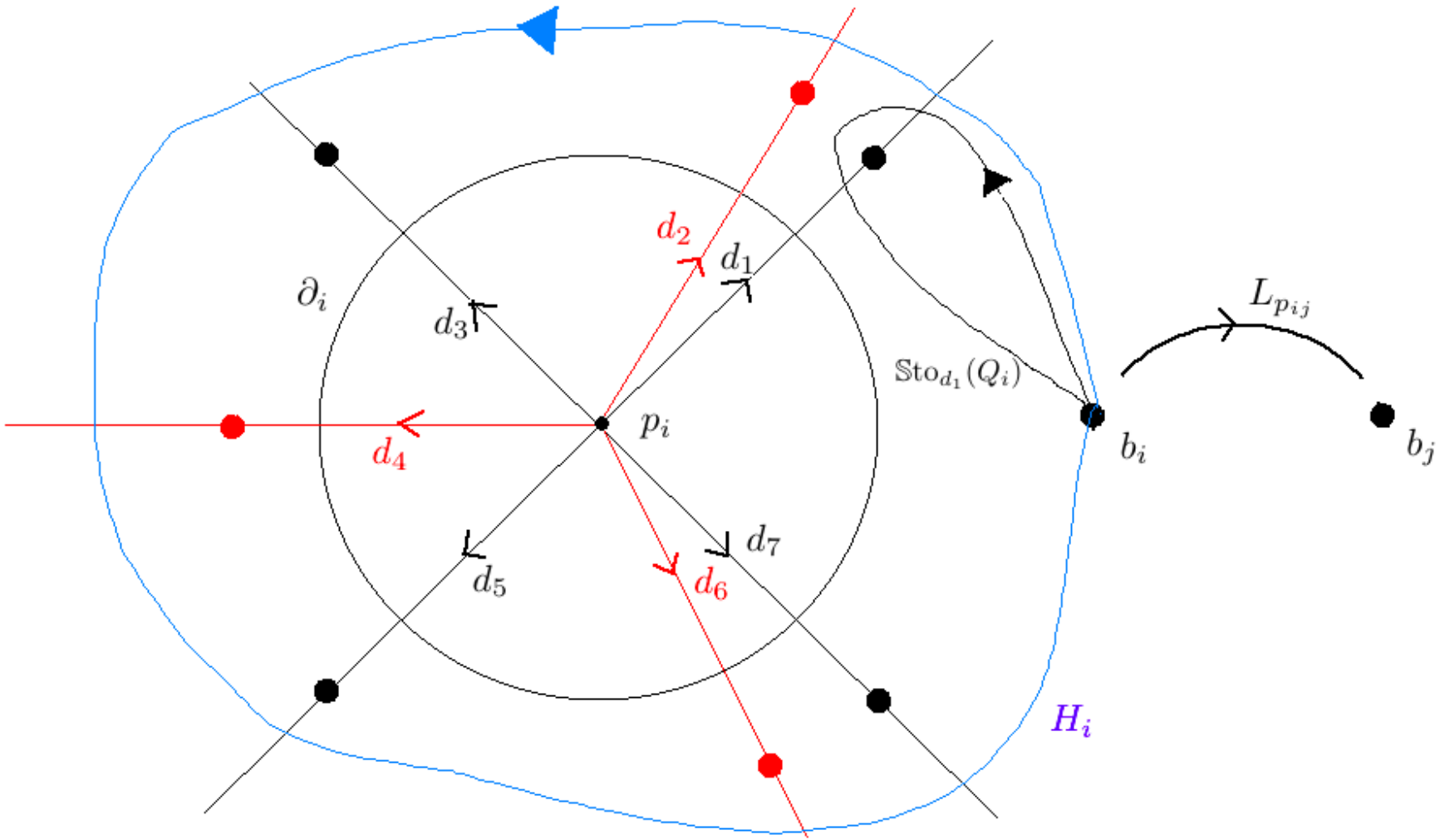
Stokes representation: representation $\rho$ of $\Pi$ in $G$, satisfying:

- on a loop based at $b_i$ winding around a puncture corresponding to a singular direction $d$, $\rho$ takes values in $\mathbb{S}\mathrm{to}_d(Q_i)$,
- on a loop based at base point $b_i$ around $p_i$, $\rho$ takes values in $H_i$.

## Theorem (P. Boalch'14)

*The space* $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ *of Stokes representations of* $\Pi$ *in* $G$ *is a smooth complex affine variety and is canonically a quasi-Hamiltonian* $H$*-space, where* $H = H_1 \times \cdots \times H_m \subset G^m$.

# Stokes representations for JKTII



# RHB correspondence

**Theorem (Riemann–Hilbert–Birkhoff correspondence, P. Boalch'14)**

*The isomorphism classes of meromorphic connections on algebraic principal $G$-bundles on $X$ with irregular type $Q_i$ at $p_i$ correspond bijectively to the $H_1 \times \cdots \times H_m$-orbits in* $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$.

The subset of connections whose associated Stokes representation is irreducible carries a natural structure of regular holomorphic Poisson variety, called the wild character variety.

# Betti symplectic leaves

Any symplectic leaf of the wild character variety is given as the quasi-Hamiltonian reduction of $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ at $\prod_i \mathcal{C}_i$ for some set of conjugacy classes $\mathcal{C}_i \subset H_i$:

$$\mu_H^{-1}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_m)/H_1 \times \cdots \times H_m,$$

where

$$\mu_H \colon \mathrm{Hom}_{\mathbb{S}}(\Pi, G) \to H_1 \times \cdots \times H_m$$
$$\{(C_i, h_i, S_1^i, S_2^i, \ldots)\}_{i=1}^m \mapsto (h_1, \ldots, h_m).$$

The wild character variety determined by $G$, the irregular curve $(\mathbb{C}P^1, \mathbf{p}, \{Q_i\}_{i=1}^m)$ and the conjugacy classes $\{\mathcal{C}_i\}_{i=1}^m$ is the holomorphic symplectic variety above.

# RHB for symplectic leaves

For fixed $\Lambda_i \in \mathfrak{gl}(r,\mathbb{C})$, we let $\mathcal{C}_i$ be the conjugacy class of $\exp(2\pi\sqrt{-1}\Lambda_i)$.

**Theorem (RHB, Symplectic version)**

*The wild character variety determined by $G$, the irregular curve $(\mathbb{C}P^1, \mathbf{p}, \{Q_i\}_{i=1}^m)$ and the conjugacy classes $\{\mathcal{C}_i\}_{i=1}^m$ is $\mathbb{C}$-analytically isomorphic to the moduli space $\mathcal{M}_{dR}$ parameterizing gauge-equivalence classes of (irreducible) irregular connections having the same irregular types $Q_i$ and residues belonging to the adjoint orbit of $\Lambda_i$ at $p_i$.*

The analytical monodromy about singular point $p$ is

$$M_p = L_p H \left( \prod_{\varphi} S_\varphi \right) L_p^{-1},$$

where $\varphi$ ranges through the set of Stokes directions in positive

# Stokes directions

Consider the Stokes local systems associated to the *JKT*-systems. Let the eigenvalues of the irregular part of $\nabla$ at $\infty$ be (with $d = 2$ or 3):

$$q_i = \lambda_1 w^{-1/d} + \cdots + \lambda_k w^{-k/d}, \quad q_j = \lambda'_1 w^{-1/d} + \cdots + \lambda'_k w^{-k/d}$$

Consider their difference:

$$q_i - q_j = (\lambda_1 - \lambda'_1) w^{-1/d} + \ldots + (\lambda_l - \lambda'_l) w^{-l/d}$$

for some $k, l, d \in \mathbb{Z}_+$, so that $(\lambda_l - \lambda'_l) \neq 0$, and at least one of $\lambda_k, \lambda'_k$ is nonzero.

Stokes directions for the pair $\{q_i - q_j\}$ form the set of $\varphi \in S^1$ that are tangent at 0 to the smooth components of the curve

$$\mathrm{Re}\left((\lambda_l - \lambda'_l) w^{-l/d}\right) = 0, \qquad w = |w| e^{\sqrt{-1}\varphi}.$$

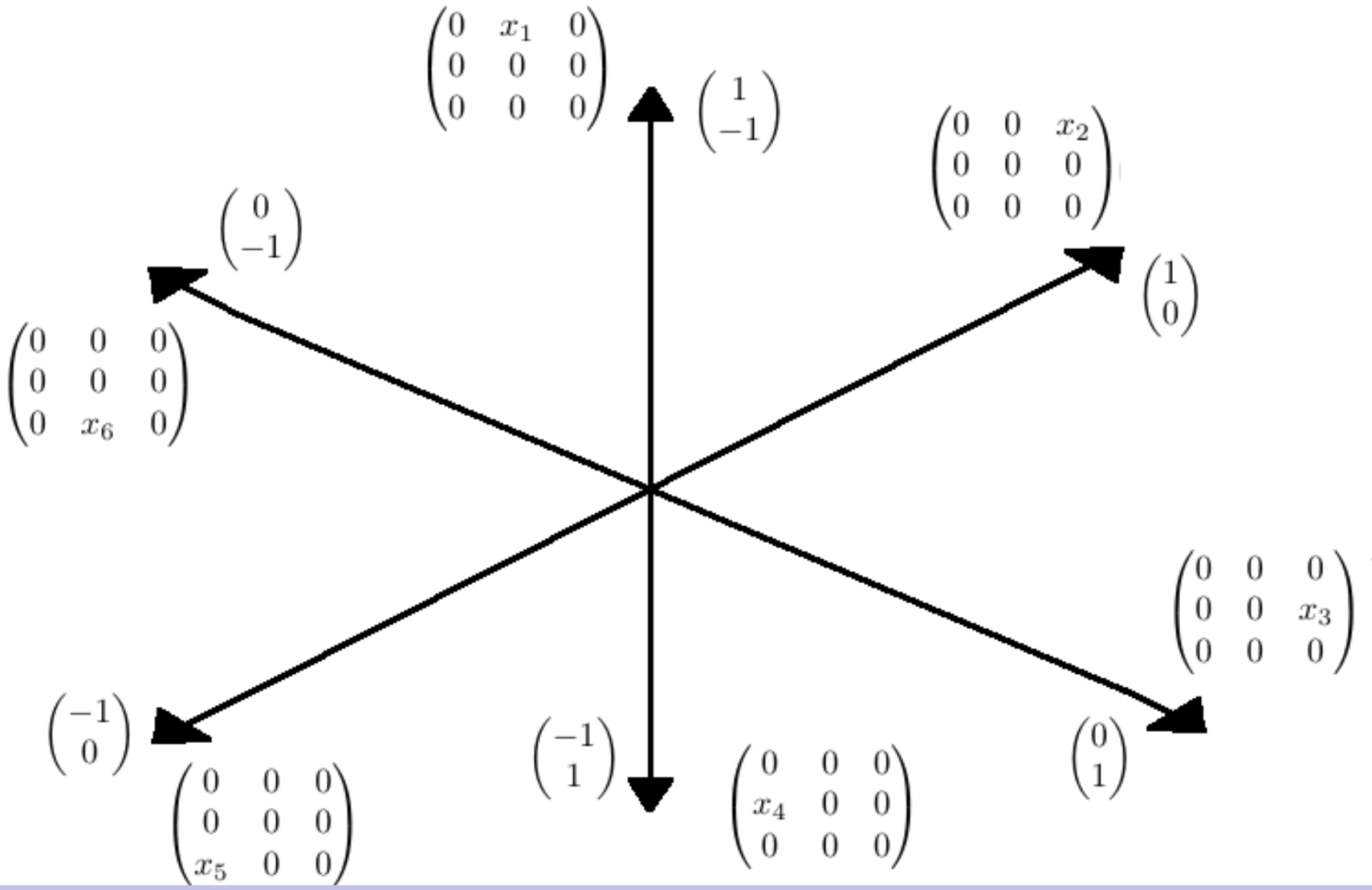
# Invariant monomials

Consider the conjugation action of the maximal torus of $G$ on matrices:

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha & x_1 & x_2 \\ x_4 & \beta & x_3 \\ x_5 & x_6 & \gamma \end{pmatrix} \cdot \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \mu^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \lambda\mu^{-1}x_1 & \lambda x_2 \\ \lambda^{-1}\mu x_4 & \beta & \mu x_3 \\ \lambda^{-1}x_5 & \mu^{-1}x_6 & \gamma \end{pmatrix}$$

The ring of invariant elements of the coordinate ring of Stokes data is generated by the monomials where the sum of the weight vectors add up to 0. These are: $x_1x_4$, $x_2x_5$, $x_3x_6$, $x_1x_3x_5$, $x_2x_4x_6$.

# $A_2$ root system with the weight vectors



# Example - *JKTII*

In the minimally twisted case, the exponential torus is 1-dimensional, i.e. the possible base changes are given by $\{e_+, e_-, e_2\} \leftrightarrow \{\lambda e_+, \lambda e_-, \lambda^{-2} e_2\}$ for $\lambda \in \mathbb{C}^\times$:

$$H_2 = \begin{pmatrix} 0 & -\alpha^{-1} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

for $\alpha \in \mathbb{C}^\times$. The ring of invariant monomials is

$$\mathbb{C}[x_1, x_2, x_3, x_5, x_6]^{H_2} = \mathbb{C}[U, V, W, R, T]/\mathcal{I}$$

where $U := x_2 x_5$, $V := x_3 x_6$, $W := x_1$, $R := x_2 x_6$, $T := x_1 x_3 x_5$, and $\mathcal{I} = (UVW - RT)$.

# Example - *JKTII*

$q_2 = -\frac{1}{2}a_1 w^{-2} + O(w^{-1})$ and
$q_{0,1} = -\frac{1}{2}a_0 w^{-2} + \lambda_3 w^{-3/2} + O(w^{-1})$
The Stokes directions are:

- $\varphi = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$, solving $\mathrm{Re}(w^{-3/2}) = 0$, corresponding to $(q_0, q_1)$ pairs,
- $\varphi = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$, solving $\mathrm{Re}(w^{-2}) = 0$, corresponding to $(q_{0,1}, q_2)$ pairs.

Thus, the Stokes matrices are in the appropriate order:

$$S_1 = \begin{pmatrix} 1 & 0 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix}, S_2 = \begin{pmatrix} 1 & x_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_5 & x_6 & 1 \end{pmatrix}, S_4 = \begin{pmatrix} 1 & 0 & 0 \\ x_4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$S_5 = \begin{pmatrix} 1 & 0 & x_8 \\ 0 & 1 & x_9 \\ 0 & 0 & 1 \end{pmatrix}, S_6 = \begin{pmatrix} 1 & x_7 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, S_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_{11} & x_{12} & 1 \end{pmatrix}.$$

# Example - *JKTII*

For the topological monodromy $M_\infty = H\prod_{i=7}^{1} S_i = I$, or equivalently $S_3S_2S_1 = (HS_7S_6S_5S_4)^{-1}$, leading to the equations

$$\alpha^{-1} = x_2x_5 + x_3x_6 + x_1x_3x_5 + 1 = U + V + T + 1$$
$$W = \alpha(UW + R + TW + VW) + 1$$
$$UVW = RT$$

After linear transformations $U = X - 1$, $V = \alpha^{-1}Y - 1$, $W = Z$, this yields the equation of the wild character variety as an affine algebraic surface:

$$\mathcal{M}_B^{JKTII} = \{(X, Y, Z) \in \mathbb{C}^3 | XYZ - X - \alpha^{-1}Y - Z + 1 + \alpha^{-1} = 0\}.$$

Agrees with the result of M. van der Put, M. Saito'09 describing the rank 2 JMU representations of the Painlevé II system.

# The main result I.

## Theorem

*Under suitable choices of parameters, the affine cubic surfaces describing the JKT wild character varieties are isomorphic to those of the corresponding Painlevé case.*
*Moreover there are concrete correspondences with the parameters of the Painlevé wild character varieties.*

| Summary | | |
|---|---|---|
| *JKT* system | equation of $\mathcal{M}_B$ moduli space | dim$\mathcal{P}$ |
| *JKTVI* | $\gamma XYZ + \alpha X^2 + \beta Y^2 + \gamma Z^2 + c_1 X + c_2 Y + c_3 Z + c_4 = 0$ | 4 |
| *JKTV* | $XYZ + X^2 + Y^2 + c_1 X + c_2 Y + c_3 Z + c_4 = 0$ | 3 |
| *JKTIVa* | $XYZ + X^2 + c_1 X + c_2 Y + c_3 Z + c_4 = 0$ | 2 |
| *JKTIVb* | $XYZ + Y^2 + c_1 X + c_2 Y + c_3 Z + c_4 = 0$ | 2 |
| *JKTII* | $XYZ - X - \alpha^{-1} Y - Z + 1 + \alpha^{-1} = 0$ | 1 |
| *JKTI* | $XYZ + X + Y + 1 = 0$ | 0 |

# The main result II.

**Theorem**

*Fourier–Laplace transformation* $\mathcal{F}$ *of* $\mathcal{D}$*-modules induces a hyperKähler isometry between the moduli spaces of rank* 2 *irregular connections associated to the Painlevé equations and the corresponding rank* 3 *JKT moduli spaces of irregular connections.*

- Strategy: show that $\mathcal{F}$ is algebraic in both the Dolbeault and de Rham complex structures, and preserves the Dolbeault holomorphic symplectic form.
- J. Douçot'24 analyzed the de Rham parts of the same isomorphisms from the perspective of supernova graphs.
- P. Boalch'12 described more generally certain isomorphisms between moduli spaces of irregular connections defined over the trivial bundle given by different "readings" of so-called supernova graphs.

# (Geometric) Fourier–Laplace transformation

- Let $\widehat{X} = \operatorname{Spec} \mathbb{C}[\hat{z}] \cup \operatorname{Spec} \mathbb{C}[\hat{w}]$ be another copy of $\mathbb{C}P^1$ ($\hat{w} = \hat{z}^{-1}$).
- Denote the projection morphisms by $p\colon X \times \widehat{X} \to X$, $\hat{p}\colon X \times \widehat{X} \to \widehat{X}$.
- Denote by $\psi$ the sheaf of $\mathcal{D}_{X\times\widehat{X}}$-modules that is the free $\mathcal{O}_{X\times\widehat{X}}$-module of rank 1, endowed with the connection

$$\mathrm{d} + \mathrm{d}(z\hat{z}) = \mathrm{d} + \hat{z}\,\mathrm{d}z + z\,\mathrm{d}\hat{z}.$$

- A minimal extension $\mathbb{M}_{\min}$ of $\mathbb{M}$ over $C$ is defined by
  1. $\mathbb{C}(z) \otimes_{\mathbb{C}[z]} \mathbb{M}_{\min} \cong \mathbb{C}(z) \otimes_{\mathbb{C}[z]} \mathbb{M}$ (it is an extension), and
  2. $\mathbb{M}_{\min}$ admits no quotient or submodules supported in dimension 0 (it is minimal).

**Definition**

The Fourier–Laplace transform of $\mathbb{M}$ is the $\mathcal{D}_{\widehat{X}}$-module

$$\widehat{\mathbb{M}} = \mathcal{F}(\mathbb{M}) = R\hat{p}_*(p^*\mathbb{M}_{\min} \otimes \psi)[1]$$

# Local Fourier–Laplace transformation

Aim: understand the contribution of the singularities of $\mathbb{M}$ at its singular points $C$ to the singularities of its Fourier transform $\widehat{\mathbb{M}}$. We use the definition of G. Laumon'87 of local versions $\mathcal{F}^{0,\widehat{\infty}}, \mathcal{F}^{\infty,\hat{0}}, \mathcal{F}^{\infty,\widehat{\infty}}$ of the functor $\mathcal{F}$ for $\ell$-adic sheaves. B. Malgrange'91: gave $\mathcal{D}$-module versions.

**Theorem (G. Laumon, B. Malgrange)**

*The functors $\mathcal{F}^{0,\widehat{\infty}}, \mathcal{F}^{\infty,\hat{0}}, \mathcal{F}^{\infty,\widehat{\infty}}$ are exact. Furthermore, assume that $\mathbb{M}$ is a holonomic $\mathcal{D}_X$-module with singular set $C \subseteq \{0, \infty\}$, and if $0 \in C$ then it is a regular singularity, and $\infty$ is an irregular singularity. By the corresponding data of $\mathbb{M}$, the followings can be completely described:*

- *Swan-conductor at a singular point and the rank of $\widehat{\mathbb{M}}$*
- *list of singularities of $\widehat{\mathbb{M}}$ and the formal types of $\widehat{\mathbb{M}}$ at its singular points*

# Example - *JKTII*

The only (irregular) singularity of $\mathbb{M}$ being placed at $\infty$, we choose

$$A'_{-3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q \end{pmatrix}, \quad A'_{-2} = \begin{pmatrix} q_1 & 0 & 0 \\ q_3 & q_1 & 0 \\ 0 & 0 & q_2 \end{pmatrix}$$

for $q, q_3 \neq 0$ and arbitrary $q_1, q_2$. Then, $\widehat{\mathbb{M}}$ has a regular singularity at $\hat{z} = q_2$. Moreover, it has an irregular singularity at $\hat{z} = \widehat{\infty}$ with

$$\mathsf{Sw}(\widehat{\mathbb{M}}_{\widehat{\infty}}) = \mathsf{Sw}(\mathbb{M}_{\infty}) = 2 \cdot \frac{3}{2} + 2 = 5, \quad \mathsf{rk}(\widehat{\mathbb{M}}_{\widehat{\infty}}) = 5 - 3 = 2.$$

# Example - *JKTII*

Formally, $\widehat{\mathbb{M}}_{\widehat{\infty}}$ decomposes into a sum of two rank 1 modules, of slopes 3 and 2 respectively, so that

$$\hat{Q} = \hat{A}_{-4}\hat{w}^{-3} + \hat{A}_{-3}\hat{w}^{-2} + \hat{A}_{-2}\hat{w}^{-1}$$

with

$$\hat{A}_{-4} = \begin{pmatrix} q_3^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{A}_{-3} = \begin{pmatrix} * & 0 \\ 0 & -q^{-1} \end{pmatrix}.$$

This gives a special case of the JMU representation of Painlevé II, namely one eigenvalue of $\hat{A}_{-4}$ vanishes.

# Dolbeault spaces

- We construct a surface $Z^*$ for each $* \in \{VI, V, IVa, IVb, II, I\}$ such that the Beauville–Narasimhan–Ramanan (or BNR or spectral) correspondence establishes an algebraic isomorphism between $\mathcal{M}_{\mathrm{Dol}}^{JKT*}$ and the moduli space $\mathcal{M}_{\mathrm{Muk}}^{*}$ of sheaves on $Z^*$ with suitably fixed Chern classes.
- The surface $Z^*$ can be defined as a suitable multiple blow-up of the Hirzebruch surface of index one $\mathbb{F}_1$, followed by the blow-down of its section at $\infty$:

$$\omega \colon Z^* \dashrightarrow \mathbb{F}_1.$$

# Spectral description

Consider $K_{\mathbb{C}P^1}(D) \cong \mathcal{O}_{\mathbb{C}P^1}(1)$, and $p\colon \mathbb{F}_1 \to \mathbb{C}P^1$ (the ruling).
$\mathcal{O}_{\mathbb{F}_1|\mathbb{C}P^1}(1)$ the relative ample bundle of $p$, with canonical sections:

$$\zeta \in H^0(\mathbb{F}_1, p^* K_{\mathbb{C}P^1}(D) \otimes \mathcal{O}_{\mathbb{F}_1|\mathbb{C}P^1}(1)), \quad \xi \in H^0(\mathbb{F}_1, \mathcal{O}_{\mathbb{F}_1|\mathbb{C}P^1}(1)).$$

The section $\xi = 0$ is denoted by $\sigma_\infty$: section at infinity of $p$.

## Definition

The *spectral curve* $\Sigma_{(\mathcal{E},\theta)}$ of $(\mathcal{E}, \theta)$ is the algebraic curve determined by:

$$\chi_\theta(\zeta, \xi) = \det(\zeta I_{\mathcal{E}} - \xi p^*\theta) = \zeta^3 + F_\theta \zeta^2 \xi + G_\theta \zeta \xi^2 + H_\theta \xi^3 = 0.$$

The *spectral sheaf* is the coherent sheaf $\mathcal{S}_{(\mathcal{E},\theta)}$ on $\mathbb{F}_1$ defined by:

$$0 \to p^*(\mathcal{E} \otimes K_{\mathbb{C}P^1}(D)^{-1}) \xrightarrow{\zeta I_{\mathcal{E}} - \xi p^*\theta} p^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{F}_1|\mathbb{C}P^1}(1) \to \mathcal{S}_{(\mathcal{E},\theta)} \to 0.$$

# Hitchin fibration

The characteristic coefficients $F_\theta, G_\theta, H_\theta$ satisfy:

$$(F_\theta, G_\theta, H_\theta) \in H^0(\mathbb{C}P^1, K_{\mathbb{C}P^1}(D)) \oplus H^0(\mathbb{C}P^1, K_{\mathbb{C}P^1}(D)^{\otimes 2}) \oplus H^0(\mathbb{C}P^1, K_{\mathbb{C}P^1}(D)^{\otimes 3}) = \mathcal{B}$$

$\mathcal{B}$ is called the Hitchin base.

## Proposition

*For fixed values of the parameters in the local forms of $\theta$, the set of possible $F_\theta, G_\theta, H_\theta$ forms an affine subspace of $\mathcal{B}$ of dimension 1.*

## Definition

The map $h$, called the *Hitchin fibration* is

$$\begin{aligned} h : \mathcal{M}_{Dol}^{JKT*} &\to \mathbb{C} \\ (\mathcal{E}, \theta) &\mapsto (F_\theta, G_\theta, H_\theta) \end{aligned}$$

# BNR-correspondence

**Theorem (Beauville–Narasimhan–Ramanan correspondence, Sz. '17)**

*There is an equivalence of categories between the groupoid of irregular Higgs bundles with prescribed polar part and the relative Picard scheme parameterizing torsion-free sheaves of rank 1 and of given degree on holomorphic curves in a given homology class* $[F_\infty^{JKT*}] \in H_2(Z^*, \mathbb{Z})$ *on a certain multiple blow-up* $Z^*$ *of the Hirzebruch surface* $\mathbb{F}_1$.

Here, $F_\infty^{JKT*}$: fiber at infinity. As a consequence:

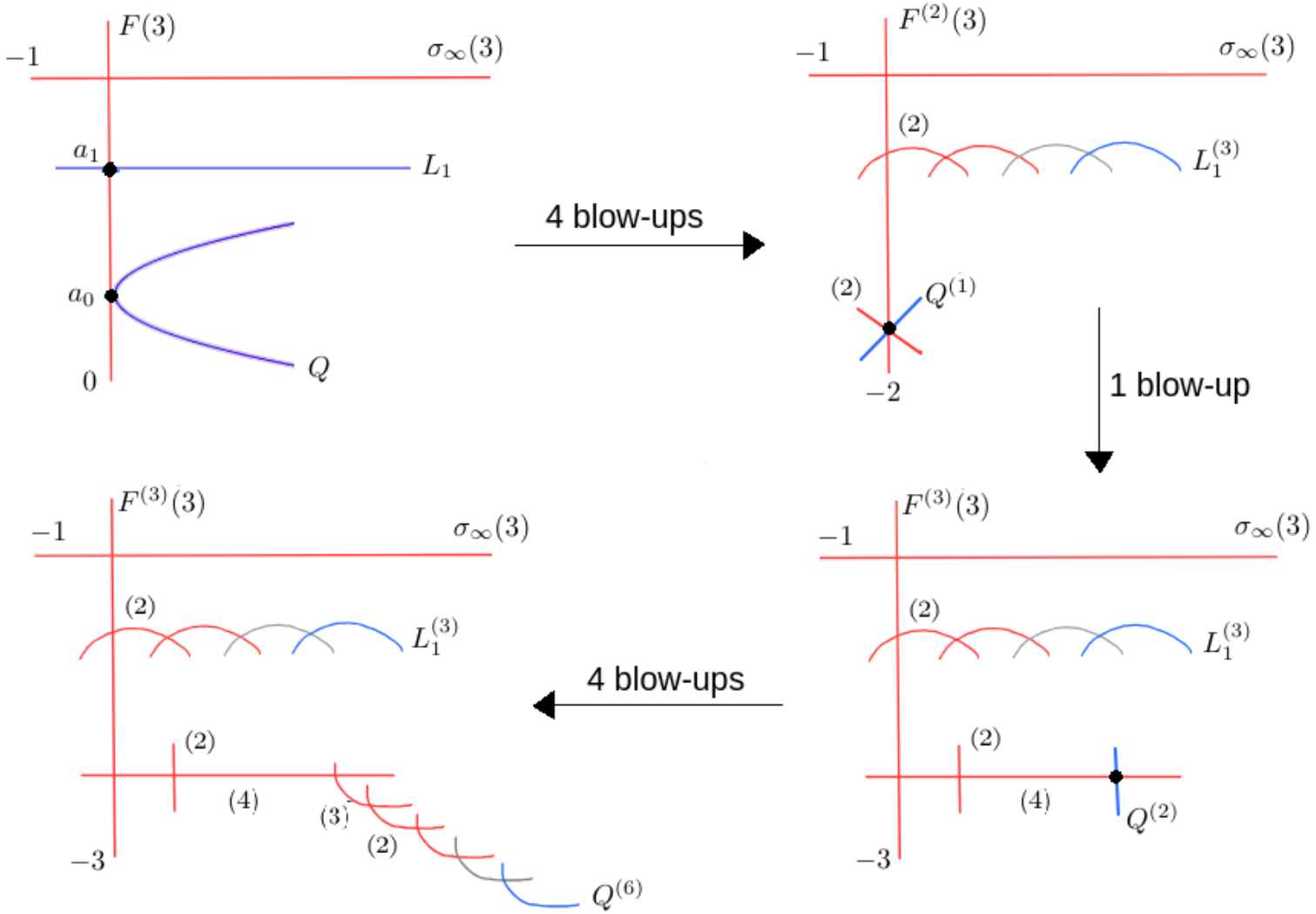
**Proposition**

*For generic* $b_0 \in \mathcal{B}$, *the fiber* $h^{-1}(b_0)$ *of the Hitchin fibration is a torsor over the Jacobian* $\mathrm{Jac}(\Sigma_{(\mathcal{E},\theta)})$ *for any* $(\mathcal{E}, \theta) \in h^{-1}(b_0)$. *In particular,* $h^{-1}(b_0)$ *is a smooth elliptic curve.*

# Example - Construction of $Z^{II}$

- Consider the curves $C_1 = Q + L_1$, and $C_2 = 3F + 3\sigma_\infty$, with corresponding divisors $D_1 = (f_1), D_2 = (f_2)$ (for some $f_i$ functions) on $\mathbb{F}_1$.
- Consider
$$\{(\lambda_1 f_1 + \lambda_2 f_2) | [\lambda_1 : \lambda_2] \in \mathbb{C}P^1\}$$
complete family of linearly equivalent divisors (i.e. a pencil).
- Blow up the pencil at its base points $C_1 \cap C_2$, and blow-down $\sigma_\infty$
- It is easy to see that the fiber $F_\infty^{JKTII}$ at infinity comes from the red curve, and it is of type $\tilde{E}_7$.
- We see in particular that $H^0(Z^*, \mathcal{O}(F_\infty^{JKTII})) \cong \mathbb{C}^2$, and $[F_\infty^{JKTII}] = -[K_{Z^*}]$.

# Example - Construction of $Z^{II}$



# Summary of the six cases

| Summary | | | |
|---|---|---|---|
| *JKT* system | polar divisor of $\theta$ | type of irregular singularity at $\infty$ | Dynkin diagram of $F_\infty^{JKT*}$ |
| *JKTVI* | $D = \{0\} + 2\{\infty\}$ | untwisted | $I_0^*$ (or $\widetilde{D}_4$) |
| *JKTV* | $D = \{0\} + 2\{\infty\}$ | minimally twisted | $I_1^*$ (or $\widetilde{D}_5$) |
| *JKTIVa* | $D = \{0\} + 2\{\infty\}$ | maximally twisted | $\widetilde{E}_6$ |
| *JKTIVb* | $D = 3\{\infty\}$ | untwisted | $\widetilde{E}_6$ |
| *JKTII* | $D = 3\{\infty\}$ | minimally twisted | $\widetilde{E}_7$ |
| *JKTI* | $D = 3\{\infty\}$ | maximally twisted | $\widetilde{E}_8$ |

# Algebraic Nahm transform

Our aim is to show:

**Theorem**

*Fourier–Laplace transformation establishes an algebraic symplectic isomorphism with respect to the Dolbeault complex structures between each $\mathcal{M}_{Dol}^{JKT*}$ and the corresponding $\mathcal{M}_{Dol}^{P*}$.*

For each $Z^*$, there exists a birational morphism $Z^* \dashrightarrow X \times \widehat{X}$ , defined on the complement of the blow-down of $\sigma_\infty$,
In particular, the projection morphisms $p\colon Z^* \dashrightarrow X$, $\hat{p}\colon Z^* \dashrightarrow \widehat{X}$ are well-defined away from the blow-down of $\sigma_\infty$.

**Definition**

The (algebraic) Nahm transform of $(\mathcal{E}, \theta) \in \mathcal{M}_{\mathrm{Dol}}^{JKT*}$ is

$$\mathcal{N}(\mathcal{E}, \theta) = (\hat{p}_* \mathcal{S}_{(\mathcal{E},\theta)}, \hat{p}_*(p^* z \cdot)\, \mathrm{d}\hat{z}).$$

# Properties of Nahm transformation

$\mathbf{\Omega}_{\mathrm{Dol}}^{JKT*} \in H^{2,0}(\mathcal{M}_{\mathrm{Dol}}^{JKT*})$, and $\mathbf{\Omega}_{\mathrm{Dol}}^{P*} \in H^{2,0}(\mathcal{M}_{\mathrm{Dol}}^{P*})$: holomorphic symplectic 2-forms.

## Proposition

*Nahm transformation is an algebraic symplectic isomorphism*

$$\mathcal{N}\colon (\mathcal{M}_{Dol}^{JKT*}, \mathbf{\Omega}_{Dol}^{JKT*}) \to (\mathcal{M}_{Dol}^{P*}, \mathbf{\Omega}_{Dol}^{P*}).$$

*Moreover, there exists a commutative diagram of diffeomorphisms between smooth 4-manifolds:*

$$\begin{array}{ccc} \mathcal{M}_{Dol}^{JKT*} & \xrightarrow{\mathcal{N}} & \mathcal{M}_{Dol}^{P*} \\ \Big\downarrow{\scriptstyle NAH} & & \Big\downarrow{\scriptstyle NAH} \\ \mathcal{M}_{dR}^{JKT*} & \xrightarrow{\mathcal{F}} & \mathcal{M}_{dR}^{P*} \end{array}$$

# Mukai moduli spaces and conclusion

- Consider the log-Calabi–Yau pairs $(Z^*, F_\infty^{JKT*})$.
- We let $\mathcal{M}^*_{\mathrm{Muk}}$ denote the moduli space of simple coherent sheaves $\mathcal{S}$ on $Z^*$ such that
  - $c_0(\mathcal{S}) = 0 \in H^0(Z^*, \mathbb{Z})$,
  - $c_1(\mathcal{S}) = PD[F_\infty^{JKTII}] \in H^2(Z^*, \mathbb{Z})$,
  - $c_2(\mathcal{S}) = PD[pt] \in H^4(Z^*, \mathbb{Z})$,

  ($PD$=Poincaré duality, and $[pt]$=homology class of a point)
- It is a smooth symplectic quasi-projective surface with holomorphic symplectic form $\mathbf{\Omega}^*_{\mathrm{Muk}} \in H^{2,0}(\mathcal{M}^*_{\mathrm{Muk}})$.

## Proposition

*The isomorphism between $\mathcal{M}^{JKT*}_{Dol}$ and $\mathcal{M}^*_{Muk}$ preserves their holomorphic symplectic 2-forms $\mathbf{\Omega}^{JKT*}_{Dol}$ and $\mathbf{\Omega}^*_{Muk}$, and the same holds for the isomorphism between $\mathcal{M}^{P*}_{Dol}$ and $\mathcal{M}^*_{Muk}$ too.*

Thank you for your attention!